

Delay equations with non-negativity constraints driven by a Hölder continuous function of order

$$\beta \in (\frac{1}{3}, \frac{1}{2})$$

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Abstract

In this note we prove an existence and uniqueness result of solution for multidimensional delay differential equations with normal reflection and driven by a Hölder continuous function of order $\beta \in (\frac{1}{3}, \frac{1}{2})$. We also obtain a bound for the supremum norm of this solution. As an application, we get these results for stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$.

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Running head: delay equations with constraints

1 Introduction

The theory of rough path analysis has been developed from the initial paper by Lyons [14]. The aim of this theory is to analyze dynamical systems $dx_t = f(x_t)dy_t$, where the control function y is not differentiable but has finite p -variation for some $p > 1$. There is a wide literature on rough path analysis (see, for instance, Lyons and Qian [16], Friz and Victoir [6], Lejay [13], Lyons [15] or Gubinelli [9]).

A path-wise approach to classical stochastic calculus has been one of the motivations to build rough path analysis theory. A nice application of the rough path analysis is the stochastic calculus with respect to the fractional Brownian motion with Hurst parameter $H \in (0, 1)$. We refer, for instance to Coutin and Lejay [3], Friz and Victoir [8], Friz [7] and Ledoux *et al.* [17] for some applications of rough path analysis to the stochastic calculus.

Nualart and Răşcanu in [19] developed an alternative approach to the study of dynamical systems $dx_t = f(x_t)dy_t$, where the control function y is Hölder continuous of order $\beta > \frac{1}{2}$. In this case the Riemann-Stieltjes integral $\int_0^t f(x_s)dy_s$ can be expressed as a Lebesgue integral using fractional derivatives following the ideas of Zähle [22]. Later, Hu and Nualart [10] extended this approach to the case $\beta \in (\frac{1}{3}, \frac{1}{2})$. In this work they give an explicit expression for the integral $\int_0^t f(x_s)dy_s$ that depends on the functions x , y and a quadratic multiplicative functional $x \otimes y$. Using this formula, the authors have established the existence and uniqueness of a solution for the dynamical system $dx_t = f(x_t)dy_t$ driven by a Hölder continuous function y of order $\beta \in (\frac{1}{3}, \frac{1}{2})$. Finally, using the same approach, Besalú and Nualart [1] got estimates for the supremum norm of the solution.

The purpose of this paper is to study a differential delay equation with non-negativity constraints driven by a Hölder continuous function y of order $\beta \in (\frac{1}{3}, \frac{1}{2})$ using the methodology introduced in [10]. We will consider the problems of existence, uniqueness and boundedness of the solutions. As an application we will study a stochastic delay differential equations with non-negativity constraints driven by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$. These results extend the work by Besalú and Rovira [2], where is considered the case $H > \frac{1}{2}$.

More precisely, we consider a delay differential equation with positivity constraints on \mathbb{R}^d of the form:

$$\begin{aligned} x(t) &= \eta(0) + \int_0^t b(s, x) ds + \int_0^t \sigma(x(s-r)) dy_s + z(t), \quad t \in (0, T], \\ x(t) &= \eta(t), \quad t \in [-r, 0], \end{aligned}$$

where r denotes a strictly positive time delay, y is a m -dimensional β -Hölder continuous function with $\frac{1}{3} < \beta < \frac{1}{2}$, $b(s, x)$ the hereditary term, depends on the path $\{x(u), -r \leq u \leq s\}$, while $\eta : [-r, 0] \rightarrow \mathbb{R}_+^d$ is a non negative smooth function, with $\mathbb{R}_+^d = \{u \in \mathbb{R}^d; u_i \geq 0 \text{ for } i = 1, \dots, d\}$ and z is a vector-valued non-decreasing process which ensures that the non-negativity constraints on x are enforced.

Then, we will apply pathwise our deterministic result to a stochastic delay differential equation with positivity constraints on \mathbb{R}^d of the form:

$$\begin{aligned} X(t) &= \eta(0) + \int_0^t b(s, X) ds + \int_0^t \sigma(X(s-r)) dW_s^H + Z(t), \quad t \in (0, T], \\ X(t) &= \eta(t), \quad t \in [-r, 0], \end{aligned}$$

where $W^H = \{W^{H,j}, j = 1, \dots, m\}$ are independent fractional Brownian motions with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, while $\eta : [-r, 0] \rightarrow \mathbb{R}_+^d$ is a deterministic non negative smooth function and Z is a vector-valued non-decreasing process which ensures that the non-negativity constraints on X are enforced.

As far as we know, stochastic delay differential equations with constraints and driven by a fractional Brownian motion has only been considered when $H > \frac{1}{2}$ ([2]). Furthermore, the literature about stochastic delay differential equations driven by a fractional Brownian motion is scarce. For the case $H > \frac{1}{2}$ has been studied the existence and uniqueness of solution ([4], [12]), the existence and regularity of the density ([12]) and the convergence when the delay goes to zero ([5]). For $H < \frac{1}{2}$ we can find the results about the existence and uniqueness of solution ([18], [21]). Actually, in [18] the authors consider a similar equation to our case but without reflection. Moreover, they use another approach in order to define the stochastic integral based on Lévy area. In any case, we will use some results on fractional Brownian motion taken from this paper.

Anyway, as it has been described in this paper of Kinnally and Williams [11] there are some models affected by some type of noise where the dynamics are related to propagation delay and some of them are naturally non-negative quantities. So, it is natural to continue the study of the stochastic delay differential equations and non-negativity constraints driven by a fractional Brownian motion.

In our work, we will make use of the techniques introduced by Hu and Nualart [10] with some ideas borrowed from Besalú and Rovira [2]. In this framework, let us point out again that one novelty of our paper is the non-negative constraints dealing with equations driven by a Hölder

continuous function of order $\beta \in (\frac{1}{3}, \frac{1}{2})$. We have used the Skorohod's mapping. Let us recall now the Skorokhod problem. Set

$$\mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d) := \{x \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) : x(0) \in \mathbb{R}_+^d\}.$$

Definition 1.1 *Given a path $z \in \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d)$, we say that a pair (x, y) of functions in $\mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d)$ solves the Skorokhod problem for z with reflection if*

1. $x(t) = z(t) + y(t)$ for all $t \geq 0$ and $x(t) \in \mathbb{R}_+^d$ for each $t \geq 0$,
2. for each $i = 1, \dots, d$, $y^i(0) = 0$ and y^i is nondecreasing,
3. for each $i = 1, \dots, d$, $\int_0^t x^i(s) dy_s^i = 0$ for all $t \geq 0$, so y^i can increase only when x^i is at zero.

It is known that we have an explicit formula for y in terms of z : for each $i = 1, \dots, d$

$$y^i(t) = \max_{s \in [0, t]} (z^i(s))^-.$$

The path z is called the reflector of x and the path y is called the regulator of x . We use the Skorokhod mapping for constraining a continuous real-valued function to be non-negative by means of reflection at the origin.

The structure of the paper is as follows: in the next section we give some preliminaries, our hypothesis and we state the main results of our paper. In Section 3, we give some basic facts about fractionals integrals. Section 4 is devoted to prove our main result: the existence and uniqueness for the solution for deterministic equations, while Section 5 deals with the problem of the boundedness. In Section 6 we apply the deterministic results to the stochastic case. Finally, Section 7 is devoted to give some technical results, as a fixed point theorem, and some properties related to the Skorohod problem.

2 Main results

Fix a time interval $[0, T]$. For any function $x : [0, T] \rightarrow \mathbb{R}^n$, the γ -Hölder norm of x on the interval $[s, t] \subset [0, T]$, where $0 < \gamma \leq 1$, will be denoted by

$$\|x\|_{\gamma(s, t)} = \sup_{s < u < v < t} \frac{|x(v) - x(u)|}{(v - u)^\gamma}.$$

If $\Delta_T := \{(s, t) : 0 \leq s < t \leq T\}$, for any $(s, t) \in \Delta_T$ and for any $g : \Delta_T \rightarrow \mathbb{R}^n$ we set

$$\|g\|_{\gamma(s, t)} = \sup_{s < u < v < t} \frac{|g(u, v)|}{(v - u)^\gamma}.$$

We will also set $\|x\|_\gamma = \|x\|_{\gamma(0, T)}$ and $\|x\|_{\gamma(r)} = \|x\|_{\gamma(-r, T)}$. Moreover, $\|\cdot\|_{\infty(s, t)}$ will denote the supremum norm in the interval (s, t) , and for simplicity $\|x\|_\infty = \|x\|_{\infty(0, T)}$ and $\|x\|_{\infty(r)} = \|x\|_{\infty(-r, T)}$.

Fix $0 < \beta \leq 1$. As in [14] we introduce the following definition.

Definition 2.1 We will say that $(x, y, x \otimes y)$ is an (d, m) -dimensional β -Hölder continuous multiplicative functional if:

1. $x : [0, T] \rightarrow \mathbb{R}^d$ and $y : [0, T] \rightarrow \mathbb{R}^m$ are β -Hölder continuous functions,
2. $x \otimes y : \Delta_T \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ is a continuous function satisfying the following properties:

(a) (Multiplicative property) For all $s \leq u \leq t$ we have

$$(x \otimes y)_{s,u} + (x \otimes y)_{u,t} + (x(u) - x(s)) \otimes (y(t) - y(u)) = (x \otimes y)_{s,t}.$$

(b) For all $(s, t) \in \Delta_T$

$$|(x \otimes y)_{s,t}| \leq c|t - s|^{2\beta}.$$

We will denote by $M_{d,m}^\beta(0, T)$ the space of (d, m) -dimensional β -Hölder continuous multiplicative functionals. Furthermore, we will denote by $M_{d,m}^\beta(a, b)$ the obvious extension of the definition $M_{d,m}^\beta(0, T)$ to a general interval (a, b) . We refer the reader to [14] and [10] for a more detailed presentation on β -Hölder continuous multiplicative functionals.

We have now the tools to give our results. Set $\beta \in (\frac{1}{3}, \frac{1}{2})$.

Consider the deterministic stochastic differential equation on \mathbb{R}^d

$$\begin{aligned} x(t) &= \eta(0) + \int_0^t b(s, x) ds + \int_0^t \sigma(x(s-r)) dy_s + z(t), \quad t \in (0, T], \\ x(t) &= \eta(t), \quad t \in [-r, 0], \end{aligned} \tag{2.1}$$

where for each $i = 1, \dots, d$,

$$z^i(t) = \max_{s \in [0, t]} (\xi^i(s))^- , \quad t \in [0, T],$$

and

$$\xi(t) = \eta(0) + \int_0^t b(s, x) ds + \int_0^t \sigma(x(s-r)) dy_s, \quad t \in [0, T].$$

Let us consider the following hypothesis:

- (H1) $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^m$ is a continuously differentiable function such that σ' is locally γ -Hölder continuous for $\gamma > \frac{1}{\beta} - 2$.
- (H2) $b : [0, T] \times C(-r, T; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ is a measurable function such that for every $t > 0$ and $f \in C(-r, T; \mathbb{R}^d)$, $b(t, f)$ depends only on $\{f(s); -r \leq s \leq t\}$. Moreover, there exists $b_0 \in L^\rho(0, t; \mathbb{R}^d)$ with $\rho \geq 2$ and $\forall N \geq 0$ there exists $L_N > 0$ such that:

$$(1) \quad |b(t, x) - b(t, y)| \leq L_N \|x - y\|_{\infty(-r, t)}, \quad \forall x, y \text{ such that } \|x\|_{\infty(r)} \leq N, \|y\|_{\infty(r)} \leq N, \quad \forall t \in [0, T],$$

$$(2) \quad |b(t, x)| \leq L_0 \|x\|_{\infty(-r, t)} + b_0(t), \quad \forall t \in [0, T].$$

The result of existence and uniqueness states as follows:

Theorem 2.2 Assume that σ and b satisfy the hypothesis (H1) and (H2) respectively with $\rho \geq \frac{1}{1-\beta}$. Assume also that $\eta \geq 0$, $(\eta_{-r}, y, \eta_{-r} \otimes y) \in M_{d,m}^\beta(0, r)$ and $(y_{-r}, y, y_{-r} \otimes y) \in M_{m,m}^\beta(r, T)$. Then the equation (2.1) has a unique solution $x \in \mathcal{C}(-r, T; \mathbb{R}_+^d)$.

Remark. If we assume that $\eta \geq 0$ is a differentiable continuous function with positive derivative, then the assumptions on η of this theorem are satisfied.

In order to study the boundedness of the solutions we need to strengthen our hypothesis. Consider now:

(H3) b and σ' are bounded function.

Then, the result is as follows:

Theorem 2.3 Assume that σ and b satisfy the hypothesis **(H1)**, **(H2)** and **(H3)**. Also assume that $\eta \geq 0$ satisfies $(\eta_{\cdot-r}, y, \eta_{\cdot-r} \otimes y) \in M_{d,m}^\beta(0, r)$ and finally that $(y_{\cdot-r}, y, y_{\cdot-r} \otimes y) \in M_{m,m}^\beta(r, T)$. Set

$$\mu = \|b\|_\infty + \|\sigma\|_\infty + \|\sigma'\|_\infty + \|\sigma'\|_\gamma.$$

Then, the solution of (2.1) is bounded as follows

$$\|x\|_\infty \leq 2 + \eta(0) + T \left\{ K \left(\|\eta\|_\beta + \|\eta_{\cdot-r} \otimes y\|_{2\beta} + \mu(d^{\frac{1}{2}} + 1) \left[\|y\|_\beta + \|y\|_\beta^2 + \|y_{\cdot-r} \otimes y\|_{2\beta} \right] \right) \right\}^{\frac{1}{\beta}}, \quad (2.2)$$

where K is a universal constant depending only on β and γ , and

$$\begin{aligned} \|\eta\|_\beta &:= \|\eta\|_{\beta(-r,0)}, \\ \|\eta_{\cdot-r} \otimes y\|_{2\beta} &:= \|\eta_{\cdot-r} \otimes y\|_{2\beta(0,r)}, \\ \|y\|_\beta &:= \|y\|_{\beta(0,T)}, \\ \|y_{\cdot-r} \otimes y\|_{2\beta} &:= \|y_{\cdot-r} \otimes y\|_{2\beta(r,T)}. \end{aligned}$$

Our last result is an application of the above theorems to stochastic delay differential equations. More precisely, let us consider a stochastic delay differential equation with positivity constraints on \mathbb{R}^d of the form:

$$\begin{aligned} X(t) &= \eta(0) + \int_0^t b(s, X) ds + \int_0^t \sigma(X(s-r)) dW_s^H + Z(t), \quad t \in (0, T], \\ X(t) &= \eta(t), \quad t \in [-r, 0], \end{aligned} \quad (2.3)$$

where $W^H = \{W^{H,j}, j = 1, \dots, m\}$ are independent fractional Brownian motions with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and for each $i = 1, \dots, d$

$$Z^i(t) = \max_{s \in [0, t]} (\Xi^i(s))^- , \quad t \in [0, T],$$

and

$$\Xi(t) = \eta(0) + \int_0^t b(s, X) ds + \int_0^t \sigma(s, X(s-r)) dW_s^H, \quad t \in [0, T].$$

Then, our result for the stochastic equation, reads as follows:

Theorem 2.4 Fix $\beta \in (\frac{1}{3}, H)$. Assume that σ and b satisfy the hypothesis **(H1)** and **(H2)** respectively with $\rho \geq \frac{1}{1-\beta}$. Assume also that η is a non-negative bounded function such that $(\eta_{\cdot-r}, W^H, \eta_{\cdot-r} \otimes W^H) \in M_{d,m}^\beta(0, r)$ almost surely. Then the equation (2.3) has a unique solution

$$X \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{C}(-r, T; \mathbb{R}_+^d)).$$

Furthermore, if **(H3)** is satisfied and $E(\|\eta_{\cdot-r} \otimes W^H\|_{2\beta(0,r)}^p) < \infty, \forall p \geq 1$, then $E(\|X\|_\infty^p) < \infty, \forall p \geq 1$.

3 Fractional integrals and derivatives

In this section we recall some definitions and results on fractional integrals. We refer the reader to [10] for a more detailed presentation.

Let $a, b \in \mathbb{R}$ with $a < b$. Let $f \in L^1(a, b)$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $t \in (a, b)$ by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

and

$$I_{b-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds,$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^{\infty} r^{\alpha-1} e^{-r} dr$ is the Euler gamma function. For any $p \geq 1$, let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) be the image of $L^p(a, b)$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$, then the Weyl derivatives are defined as

$$\begin{aligned} D_{a+}^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}} ds \right), \\ D_{b-}^{\alpha} f(t) &= \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t)-f(s)}{(s-t)^{\alpha+1}} ds \right), \end{aligned}$$

where $a \leq t \leq b$ (the convergence of the integrals at the singularity $s = t$ holds point-wise for almost all $t \in (a, b)$ if $p = 1$ and moreover in the L^p -sense if $1 < p < \infty$).

If $f \in C^{\lambda}(a, b)$ and $g \in C^{\mu}(a, b)$ with $\lambda + \mu > 1$, it is proved in [22] that the Riemann-Stieltjes integral $\int_a^b f dg$ exists. The following proposition provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives (see [22]).

Proposition 3.1 *Suppose that $f \in C^{\lambda}(a, b)$ and $g \in C^{\mu}(a, b)$ with $\lambda + \mu > 1$. Let $1 - \mu < \alpha < \lambda$. Then the Riemann-Stieltjes integral $\int_a^b f dg$ exists and it can be expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(t) D_{b-}^{1-\alpha} g_{b-}(t) dt, \quad (3.1)$$

where $g_{b-}(t) = g(t) - g(b)$.

But if $x, y \in C^{\beta}(a, b)$ with $\beta \in (\frac{1}{3}, \frac{1}{2})$ we can not use Equation (3.1) to define the integral $\int_a^b f(x(t)) dy_t$, so we need to recall the construction of the integral $\int_a^b f(x(t)) dy_t$ given by Hu and Nualart in [10] using fractional derivatives.

Definition 3.2 *Let $(x, y, x \otimes y) \in M_{d,m}^{\beta}(0, T)$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^m \otimes \mathbb{R}^d$ be a continuously differentiable function such that f' is locally λ -Hölder continuous, where $\lambda > \frac{1}{\beta} - 2$. Fix $\alpha > 0$ such that $1 - \beta < \alpha < 2\beta$, and $\alpha < \frac{\lambda\beta+1}{2}$. Then, for any $0 \leq a < b \leq T$ we define*

$$\begin{aligned} \int_a^b f(x(r)) dy_r &= (-1)^{\alpha} \sum_{j=1}^m \int_a^b \hat{D}_{a+}^{\alpha} f_j(x)(r) D_{b-}^{1-\alpha} y_{b-}^j(r) dr \\ &\quad - (-1)^{2\alpha-1} \sum_{i=1}^d \sum_{j=1}^m \int_a^b D_{a+}^{2\alpha-1} \partial_i f_j(x)(r) D_{b-}^{1-\alpha} \mathcal{D}_{b-}^{1-\alpha} (x \otimes y)^{i,j}(r) dr, \end{aligned}$$

where for $r \in (a, b)$

$$\widehat{D}_{a+}^{\alpha} f(x)(r) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x(r))}{(r-a)^{\alpha}} + \alpha \int_a^r \frac{f(x(r)) - f(x(\theta)) - \sum_{i=1}^m \partial_i f(x(\theta))(x^i(r) - x^i(\theta))}{(r-\theta)^{\alpha+1}} d\theta \right)$$

is the *compensated fractional derivative* and

$$\mathcal{D}_{b-}^{1-\alpha}(x \otimes y)(r) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{(x \otimes y)_{r,b}}{(b-r)^{1-\alpha}} + (1-\alpha) \int_r^b \frac{(x \otimes y)_{r,s}}{(s-r)^{2-\alpha}} ds \right)$$

is the extension of the fractional derivative of $x \otimes y$.

Let us finish this section recalling two propositions from [10]. In the sequel, k denotes a generic constant that may depend on the parameters β, α and γ .

Proposition 3.3 *Let $(x, y, x \otimes y)$ be in $M_{d,m}^{\beta}(0, T)$. Assume that $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is a continuous differentiable function such that f' is bounded and γ -Hölder continuous, where $\gamma > \frac{1}{\beta} - 2$. Then for any $0 \leq a < b \leq T$ we have*

$$\begin{aligned} \left\| \int f(x(r)) dy_r \right\|_{\beta(a,b)} &\leq k \|f(x(a))\| \|y\|_{\beta(a,b)} + k \Phi_{a,b,\beta}(x, y) \\ &\quad \times \left(\|f'\|_{\infty} + \|f'\|_{\gamma} \|x\|_{\beta(a,b)}^{\gamma} (b-a)^{\gamma\beta} \right) (b-a)^{\beta}, \end{aligned}$$

where

$$\Phi_{a,b,\beta}(x, y) = \|x \otimes y\|_{2\beta(a,b)} + \|x\|_{\beta(a,b)} \|y\|_{\beta(a,b)}.$$

Proposition 3.4 *Suppose that $(x, y, x \otimes y)$ and $(y, z, y \otimes z)$ belong to $M_{d,m}^{\beta}(0, T)$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$ be a continuously differentiable function such that f' is γ -Hölder continuous and bounded, where $\gamma > \frac{1}{\beta} - 2$. Fix $\alpha > 0$ such that $1 - \beta < \alpha < 2\beta, \alpha < \frac{\gamma\lambda+1}{2}$. Then the following estimate holds:*

$$\begin{aligned} \left| \int_a^b f(x(r)) d(y \otimes z)_{\cdot,b}(r) \right| &\leq k \|f(x(a))\| \Phi_{a,b,\beta}(y, z) (b-a)^{2\beta} \\ &\quad + k \left(\|f'\|_{\infty} + \|f'\|_{\gamma} \|x\|_{\beta(a,b)}^{\gamma} (b-a)^{\gamma\beta} \right) \Phi_{a,b,\beta}(x, y, z) (b-a)^{3\beta}, \end{aligned}$$

where

$$\Phi_{a,b,\beta}(x, y, z) = \|x\|_{\beta(a,b)} \|y\|_{\beta(a,b)} \|z\|_{\beta(a,b)} + \|z\|_{\beta(a,b)} \|x \otimes y\|_{2\beta(a,b)} + \|x\|_{\beta(a,b)} \|y \otimes z\|_{2\beta(a,b)}.$$

4 Existence and uniqueness for deterministic integral equations

The aim of this section is the proof of Theorem 2.2. For simplicity let us assume $T = Mr$.

Proof of Theorem 2.2: In order to prove that equation (2.1) admits a unique continuous solution on $[-r, T]$, we will use an induction argument. We shall prove that if the equation (2.1) admits a unique solution $x^{(n)}$ on $[-r, nr]$ we can prove that there is a unique solution $x^{(n+1)}$ on $[-r, (n+1)r]$. More precisely, our induction hypothesis is the following:

(**H_n**) The equation

$$\begin{aligned} x^{(n)}(t) &= \eta(0) + \int_0^t b(s, x^{(n)}) ds + \int_0^t \sigma(x^{(n-1)}(s-r)) dy_s + z^{(n)}(t), \quad t \in (0, nr], \\ x^{(n)}(t) &= \eta(t), \quad t \in [-r, 0], \end{aligned}$$

where for $i = 1, \dots, d$, $(z^{(n)})^i(t) = \max_{s \in [0, t]} ((\xi^{(n)})^i(s))^-$ with

$$\xi^{(n)}(t) = \eta(0) + \int_0^t b(s, x^{(n)})ds + \int_0^t \sigma(x^{(n-1)}(s-r))dy_s,$$

has a unique solution $x^{(n)} \in \mathcal{C}(-r, nr, \mathbb{R}_+^d)$ and moreover $(x_{\cdot, -r}^{(n)}, y, x_{\cdot, -r}^{(n)} \otimes y) \in M_{d,m}^\beta(0, (n+1)r)$.

Actually, when we want to check (\mathbf{H}_{n+1}) assuming (\mathbf{H}_n) , we can write the equation of (\mathbf{H}_{n+1}) as

$$\begin{aligned} x^{(n+1)}(t) &= \eta(0) + \int_0^t b(s, x^{(n+1)})ds + \int_0^t \sigma(x^{(n)}(s-r))dy_s + z^{(n+1)}(t), \quad t \in (0, (n+1)r], \\ x^{(n+1)}(t) &= \eta(t), \quad t \in [-r, 0]. \end{aligned} \quad (4.1)$$

Since $(x_{\cdot, -r}^{(n)}, y, x_{\cdot, -r}^{(n)} \otimes y) \in M_{d,m}^\beta(0, (n+1)r)$ we know that we can use Definition 3.2 to define the integral $\int_0^t \sigma(x^{(n)}(s-r))dy_s$ appearing in equation (4.1). Then, the proof will consist in checking the following steps:

1. Existence of a solution of the equation (4.1) in the space $\mathcal{C}(-r, (n+1)r; \mathbb{R}_+^d)$.
2. Uniqueness of a solution of the equation (4.1) in the space $\mathcal{C}(-r, (n+1)r; \mathbb{R}_+^d)$.
3. The solution $x^{(n+1)}$ satisfies that $(x_{\cdot, -r}^{(n+1)}, y, x_{\cdot, -r}^{(n+1)} \otimes y) \in M_{d,m}^\beta(0, (n+2)r)$.

Actually, we will only proof the first case, that is (\mathbf{H}_1) . Notice that the induction step, that is the proof of (\mathbf{H}_{n+1}) assuming that (\mathbf{H}_n) is true, can be done repeating the computations of this initial case.

So, let us check (\mathbf{H}_1) . We will deal with the equation

$$\begin{aligned} x^{(1)}(t) &= \eta(0) + \int_0^t b(s, x^{(1)})ds + \int_0^t \sigma(\eta(s-r))dy_s + z^{(1)}(t), \quad t \in (0, r], \\ x^{(1)}(t) &= \eta(t), \quad t \in [-r, 0], \end{aligned} \quad (4.2)$$

where for $i = 1, \dots, d$, $(z^{(1)})^i(t) = \max_{s \in [0, t]} ((\xi^{(1)})^i(s))^-$ and

$$\xi^{(1)}(t) = \eta(0) + \int_0^t b(s, x^{(1)})ds + \int_0^t \sigma(\eta(s-r))dy_s.$$

Note that since $(\eta_{\cdot, -r}, y, \eta_{\cdot, -r} \otimes y) \in M_{d,m}^\beta(0, r)$ we can use Definition 3.2 in order to define the integral $\int_0^t \sigma(\eta(s-r))dy_s$ appearing in (4.2). The proof of this initial case will be divided en 3 steps:

1. Existence of a solution in the space $\mathcal{C}(-r, r; \mathbb{R}_+^d)$.
2. Uniqueness of a solution in the space $\mathcal{C}(-r, r; \mathbb{R}_+^d)$.
3. The solution $x^{(1)}$ satisfies that $(x_{\cdot, -r}^{(1)}, y, x_{\cdot, -r}^{(1)} \otimes y) \in M_{d,m}^\beta(0, 2r)$

To simplify the proof we will assume $d = m = 1$.

Step 1: In order to prove the existence of solution we will use Lemma 7.1, a fixed point argument on $\mathcal{C}(-r, r, \mathbb{R}_+)$.

Let us consider the operator

$$\mathcal{L} : \mathcal{C}(-r, r; \mathbb{R}_+) \rightarrow \mathcal{C}(-r, r; \mathbb{R}_+)$$

such that

$$\begin{aligned}\mathcal{L}(u)(t) &= \eta(0) + \int_0^t b(s, u)ds + \int_0^t \sigma(\eta(s-r))dy_s + z(t), \quad t \in [0, r], \\ \mathcal{L}(u)(t) &= \eta(t), \quad t \in [-r, 0].\end{aligned}$$

where setting

$$\xi(t) = \eta(0) + \int_0^t b(s, u)ds + \int_0^t \sigma(\eta(s-r))dy_s,$$

then $z(t) = \max_{s \in [0, t]} (\xi(s))^-$.

Clearly \mathcal{L} is well defined. Let us use the notation $u^* = \mathcal{L}(u)$.

Now, we need to introduce a family of norms in the space $\mathcal{C}(-r, r; \mathbb{R}_+)$. That is, for any $\lambda \geq 1$, let us consider

$$\|f\|_{\infty, \lambda(-r, r)} := \sup_{t \in [-r, r]} e^{-\lambda t} |f(t)|.$$

It is easy to check that these norms are equivalent to $\|f\|_{\infty(-r, r)}$.

Using standard arguments (see for instance [2] for similar computations) we obtain that

$$\begin{aligned}\|u^*\|_{\infty, \lambda(-r, r)} &\leq \|\eta\|_{\infty, \lambda(-r, 0)} + 2|\eta(0)| + 2 \sup_{t \in [0, r]} e^{-\lambda t} \left| \int_0^t b(s, u)ds \right| \\ &\quad + 2 \sup_{t \in [0, r]} e^{-\lambda t} \left| \int_0^t \sigma(\eta(s-r))dy_s \right|.\end{aligned}\tag{4.3}$$

We obtain easily (see again [2]) that

$$\sup_{t \in [0, r]} e^{-\lambda t} \left| \int_0^t b(s, u)ds \right| \leq \frac{L_0}{\lambda} \|u\|_{\infty, \lambda(-r, r)} + \frac{C_\rho}{\lambda^{1-\rho}} \|b_0\|_{L^\rho}.\tag{4.4}$$

It only remains the study of the term with the fractional integral. Using the bound appearing on the proof of Proposition 3.3, we get for any $\lambda \geq 1$,

$$\begin{aligned}&\sup_{t \in [0, r]} e^{-\lambda t} \left| \int_0^t \sigma(\eta(s-r))dy_s \right| \\ &\leq k|\sigma(\eta(-r))| \|y\|_{\beta(0, r)} \sup_{t \in [0, r]} e^{-\lambda t} t^\beta + \\ &\quad + k\Phi_{0, r, \beta}(\eta_{\cdot-r}, y) \left(\|\sigma'\|_\infty \sup_{t \in [0, r]} e^{-\lambda t} t^{2\beta} + \|\sigma'\|_\gamma \|\eta_{\cdot-r}\|_{\beta(0, r)}^\gamma \sup_{t \in [0, r]} e^{-\lambda t} t^{(\gamma+2)\beta} \right) \\ &\leq k|\sigma(\eta(-r))| \|y\|_{\beta(0, r)} \left(\frac{\beta}{\lambda} \right)^\beta e^{-\beta} + \\ &\quad k\Phi_{0, r, \beta}(\eta_{\cdot-r}, y) \left(\|\sigma'\|_\infty \left(\frac{2\beta}{\lambda} \right)^{2\beta} e^{-2\beta} + \|\sigma'\|_\gamma \|\eta_{\cdot-r}\|_{\beta(0, r)}^\gamma \left(\frac{(\gamma+2)\beta}{\lambda} \right)^{(\gamma+2)\beta} e^{(\gamma+2)\beta} \right) \\ &\leq C_{\beta, \gamma} \frac{1}{\lambda^\beta} \left(|\sigma(\eta(-r))| \|y\|_{\beta(0, r)} + \Phi_{0, r, \beta}(\eta_{\cdot-r}, y) \left(\|\sigma'\|_\infty + \|\sigma'\|_\gamma \|\eta_{\cdot-r}\|_{\beta(0, r)}^\gamma \right) \right),\end{aligned}\tag{4.5}$$

where in the last inequality we have used that

$$\sup_{t \in [0, r]} t^\mu e^{-\lambda t} \leq \left(\frac{\mu}{\lambda} \right)^\mu e^{-\mu}$$

and $C_{\beta, \gamma}$ is a constant depending on β and γ .

So putting together (4.3), (4.4) and (4.5) we have

$$\|u^*\|_{\infty, \lambda(-r, r)} \leq M_1(\lambda) + M_2(\lambda) \|u\|_{\infty, \lambda(-r, r)},$$

where

$$\begin{aligned} M_1(\lambda) &= \|\eta\|_{\infty, \lambda(-r, 0)} + 2|\eta(0)| + \frac{2C_\rho}{\lambda^{1-\rho}} \|b_0\|_{L^\rho} \\ &\quad + C_{\beta, \gamma} \frac{2}{\lambda^\beta} \left(|\sigma(\eta(-r))| \|y\|_{\beta(0, r)} + \Phi_{0, r, \beta}(\eta_{-r}, y) \left(\|\sigma'\|_\infty + \|\sigma'\|_\gamma \|\eta_{-r}\|_{\beta(0, r)}^\gamma \right) \right), \\ M_2(\lambda) &= 2L_0 \frac{1}{\lambda}. \end{aligned}$$

Now, we can choose $\lambda = \lambda_0$ large enough such that $M_2(\lambda_0) \leq \frac{1}{2}$. Then, $\|u\|_{\infty, \lambda_0(-r, r)} \leq 2M_1(\lambda_0)$ yields that

$$\|u^*\|_{\infty, \lambda_0(-r, r)} \leq 2M_1(\lambda_0)$$

and $\mathcal{L}(B_0) \subseteq B_0$ for

$$B_0 = \left\{ u \in \mathcal{C}(-r, r; \mathbb{R}_+^d); \|u\|_{\infty, \lambda_0(-r, r)} \leq 2M_1(\lambda_0) \right\}.$$

The first hypothesis in Lemma 7.1 is now satisfied with the metric ρ_0 associated to the norm $\|\cdot\|_{\infty, \lambda_0(-r, r)}$. To finish the proof it suffices to find a metric ρ_1 satisfying the second hypothesis in Lemma 7.1.

Notice first that if $u \in B_0$ then $\|u\|_{\infty(-r, r)} \leq 2e^{\lambda_0 r} M_1(\lambda_0) := N_0$. Consider $u, u' \in B_0$ and $\lambda \geq 1$. Then

$$\|\mathcal{L}(u) - \mathcal{L}(u')\|_{\infty, \lambda(-r, r)} \leq \sup_{t \in [0, r]} e^{-\lambda t} |\xi(t) - \xi'(t)| + \sup_{t \in [0, r]} e^{-\lambda t} |z(t) - z'(t)|. \quad (4.6)$$

From Lemma 7.2 notice that given $t \in [0, r]$ there exists $t_2 \leq t$ such that

$$|z(t) - z'(t)| \leq K_l |\xi(t_2) - \xi'(t_2)|.$$

So

$$e^{-\lambda t} |z(t) - z'(t)| \leq K_l e^{-\lambda t_2} |\xi(t_2) - \xi'(t_2)|$$

and it follows easily that

$$\sup_{t \in [0, r]} e^{-\lambda t} |z(t) - z'(t)| \leq K_l \sup_{t \in [0, r]} e^{-\lambda t} |\xi(t) - \xi'(t)|. \quad (4.7)$$

From (4.6) and (4.7) we can write

$$\begin{aligned} \|\mathcal{L}(u) - \mathcal{L}(u')\|_{\infty, \lambda(-r, r)} &\leq (1 + K_l) \sup_{t \in [0, r]} e^{-\lambda t} \int_0^t |b(s, u) - b(s, u')| ds \\ &\leq L_{N_0} (1 + K_l) \sup_{t \in [0, r]} e^{-\lambda t} \int_0^t \sup_{0 \leq v \leq s} |u(v) - u'(v)| ds \\ &\leq L_{N_0} (1 + K_l) \sup_{t \in [0, r]} \int_0^t e^{-\lambda(t-s)} e^{-\lambda s} \sup_{-r \leq v \leq s} |u(v) - u'(v)| ds \\ &\leq L_{N_0} (1 + K_l) \frac{1}{\lambda} \|u - u'\|_{\infty, \lambda(-r, r)}. \end{aligned}$$

So, choosing $\lambda = \lambda_1$ such that $\frac{L_{N_0}(1 + K_l)}{\lambda_1} \leq \frac{1}{2}$, the second hypothesis is satisfied for the metric ρ_1 associated with the norm $\|\cdot\|_{\infty, \lambda_1(-r, r)}$ and $a = \frac{1}{2}$.

Step 2: We deal now with the uniqueness problem.

Let x and x' be two solutions of (4.2) in the space $\mathcal{C}(-r, r; \mathbb{R}_+)$ and choose N large enough such that $\|x\|_{\infty(-r, r)} \leq N$ and $\|x'\|_{\infty(-r, r)} \leq N$.

For any $t \in [0, r]$,

$$\sup_{s \in [0, t]} |x(s) - x'(s)| \leq \sup_{s \in [0, t]} |\xi(s) - \xi'(s)| + \sup_{s \in [0, t]} |z(s) - z'(s)|.$$

Moreover, using Lemma 7.2 we have

$$\sup_{s \in [0, t]} |z(s) - z'(s)| \leq K_l \sup_{s \in [0, t]} |\xi(t) - \xi'(t)|.$$

So, putting together the last two inequalities we get that

$$\begin{aligned} \sup_{s \in [0, t]} |x(s) - x'(s)| &\leq (1 + K_l) \sup_{s \in [0, t]} |\xi(s) - \xi'(s)| \\ &\leq (1 + K_l) \sup_{s \in [0, t]} \left| \int_0^s (b(\tau, x) - b(\tau, x')) d\tau \right| \\ &\leq (1 + K_l) L_N \sup_{s \in [0, t]} \left| \int_0^s \sup_{0 \leq v \leq \tau} |x(v) - x'(v)| d\tau \right| \\ &\leq L_N (1 + K_l) \int_0^t \sup_{v \in [0, \tau]} |x(v) - x'(v)| d\tau. \end{aligned}$$

Applying now Gronwall's inequality, we have that for all $t \in [0, r]$

$$\sup_{s \in [0, t]} |x(s) - x'(s)| = 0.$$

So

$$\|x - x'\|_{\infty(-r, r)} = 0$$

and the uniqueness has been proved.

Step 3: We have to prove that $(x_{\cdot, -r}^{(1)}, y, x_{\cdot, -r}^{(1)} \otimes y) \in M_{1,1}^\beta(0, 2r)$.

We have to check the three conditions appearing in Definition 2.1:

1. $y : [0, 2r] \rightarrow \mathbb{R}$ is β -Hölder continuous. This condition is one of the hypothesis of our theorem.
2. $x_{\cdot, -r}^{(1)} : [0, 2r] \rightarrow \mathbb{R}$ is β -Hölder continuous.

We can write that

$$\begin{aligned} \|x_{\cdot, -r}^{(1)}\|_{\beta(0, 2r)} &= \|x^{(1)}\|_{\beta(-r, r)} = \sup_{-r \leq v \leq w \leq r} \frac{|x^{(1)}(w) - x^{(1)}(v)|}{(w - v)^\beta} \\ &\leq \sup_{-r \leq v \leq w < 0} \frac{|\eta(w) - \eta(v)|}{(w - v)^\beta} + \sup_{\substack{-r \leq v \leq 0 \\ 0 \leq w \leq r}} \frac{|x^{(1)}(w) - \eta(v)|}{(w - v)^\beta} \\ &\quad + \sup_{0 \leq v \leq w \leq r} \frac{|x^{(1)}(w) - x^{(1)}(v)|}{(w - v)^\beta}. \end{aligned}$$

Note that

$$\frac{|x^{(1)}(w) - \eta(v)|}{(w - v)^\beta} \leq \frac{|x^{(1)}(w) - \eta(0)|}{(w - 0)^\beta} + \frac{|\eta(0) - \eta(v)|}{(0 - v)^\beta}.$$

So

$$\left\|x_{\cdot, -r}^{(1)}\right\|_{\beta(0, 2r)} \leq 2 \|\eta\|_{\beta(-r, 0)} + 2 \left\|x^{(1)}\right\|_{\beta(0, r)}. \quad (4.8)$$

Moreover

$$\left\|x^{(1)}\right\|_{\beta(0, r)} \leq \left\|\int_0^\cdot b(s, x^{(1)})ds\right\|_{\beta(0, r)} + \left\|\int_0^\cdot \sigma(\eta(s-r))dy_s\right\|_{\beta(0, r)} + \left\|z^{(1)}\right\|_{\beta(0, r)}. \quad (4.9)$$

Using Lemma 7.2 we also get that

$$\left\|z^{(1)}\right\|_{\beta(0, r)} \leq \left\|\xi^{(1)}\right\|_{\beta(0, r)}. \quad (4.10)$$

Furthermore, putting together (4.8), (4.9) and (4.10) and using again Proposition 3.3 we obtain that

$$\begin{aligned} \left\|x^{(1)}\right\|_{\beta(-r, r)} &\leq 4 \|\eta\|_{\beta(-r, 0)} + 4 \left\|\int_0^\cdot b(s, x^{(1)})ds\right\|_{\beta(0, r)} + 4 \left\|\int_0^\cdot \sigma(\eta(s-r))dy_s\right\|_{\beta(0, r)} \\ &\leq 4 \|\eta\|_{\beta(-r, 0)} + 4C \left(1 + \left\|x^{(1)}\right\|_{\infty(-r, r)}\right) + 4(k|\sigma(\eta(-r))| \|\eta\|_{\beta(0, r)} \\ &\quad + \Phi_{0, r, \beta}(\eta_{\cdot, -r}, y)(\|\sigma'\|_\infty + \|\sigma'\|_\gamma \|\eta_{\cdot, -r}\|_{\beta(0, r)}^\gamma r^{\gamma\beta}))r^\beta. \end{aligned}$$

So we can conclude that $x_{\cdot, -r}^{(1)}$ is β -Hölder continuous.

3. Let us define $(x_{\cdot, -r}^{(1)} \otimes y)_{s, t}$ for $s, t \in \Delta_{2r}$. For completeness, we will give this definition for any dimensions d and m , unless we will still consider $d = m = 1$ in the proofs. For any $k \in \{1, \dots, d\}$ and $l \in \{1, \dots, m\}$, set:

- if $s, t \in [0, r]$,

$$(x_{\cdot, -r}^{(1)} \otimes y)_{s, t}^{k, l} = (\eta_{\cdot, -r} \otimes y)_{s, t}^{k, l},$$

- if $s, t \in [r, 2r]$, set

$$\begin{aligned} (x_{\cdot, -r}^{(1)} \otimes y)_{s, t}^{k, l} &= \int_s^t (y^l(t) - y^l(v))b^k(v - r, x^{(1)})dv + \sum_{j=1}^m \int_s^t \sigma_j^k(\eta(v - 2r))d(y_{\cdot, -r} \otimes y)_{\cdot, t}^{j, l}(v) \\ &\quad + \int_s^t (y^l(t) - y^l(v))d(z^{(1)})_{v-r}^k, \end{aligned}$$

- if $s \in [0, r]$ and $t \in [r, 2r]$,

$$(x_{\cdot, -r}^{(1)} \otimes y)_{s, t}^{k, l} = (\eta_{\cdot, -r} \otimes y)_{s, r}^{k, l} + (x_{\cdot, -r}^{(1)} \otimes y)_{r, t}^{k, l} + (\eta^k(0) - \eta^k(s - r)) \otimes (y^l(t) - y^l(r)).$$

Let us check that the multiplicative property (let us recall that we consider again $d = m = 1$ for simplicity) is satisfied, that is, for any $0 \leq s \leq u \leq t \leq 2r$ it holds that

$$(x_{\cdot, -r}^{(1)} \otimes y)_{s, u} + (x_{\cdot, -r}^{(1)} \otimes y)_{u, t} + (x^{(1)}(u - r) - x^{(1)}(s - r)) \otimes (y(t) - y(u)) = (x_{\cdot, -r}^{(1)} \otimes y)_{s, t}. \quad (4.11)$$

We have to distinguish several cases:

- a) Case $0 \leq s \leq u \leq t \leq r$.

Since on Δ_r it holds that

$$(x_{\cdot, -r}^{(1)}, y, x_{\cdot, -r}^{(1)} \otimes y) = (\eta_{\cdot, -r}, y, \eta_{\cdot, -r} \otimes y),$$

the multiplicative property follows from the fact that we are assuming that $(\eta_{\cdot, -r}, y, \eta_{\cdot, -r} \otimes y)$ is a β -Hölder continuous functional.

b) Case $r \leq s \leq u \leq t \leq 2r$.

Notice first that,

$$\begin{aligned}
& (x_{\cdot-r}^{(1)} \otimes y)_{s,u} + (x_{\cdot-r}^{(1)} \otimes y)_{u,t} = \int_s^u (y(u) - y(v))b(v-r, x^{(1)})dv \\
& \quad + \int_s^u \sigma(\eta(v-2r))d(y_{\cdot-r} \otimes y)_{\cdot,u}(v) + \int_s^u (y(u) - y(v))dz_{v-r}^{(1)} \\
& \quad + \int_u^t (y(t) - y(v))b(v-r, x^{(1)})dv + \int_u^t \sigma(\eta(v-2r))d(y_{\cdot-r} \otimes y)_{\cdot,t}(v) \\
& \quad + \int_u^t (y(t) - y(v))dz_{v-r}^{(1)} \\
& = \int_s^t (y(t) - y(v))b(v-r, x^{(1)})dv + \int_s^t \sigma(\eta(v-2r))d(y_{\cdot-r} \otimes y)_{\cdot,t}(v) \\
& \quad + \int_s^t (y(t) - y(v))dz_{v-r}^{(1)} \\
& \quad + (y(u) - y(t)) \left(\int_s^u b(v-r, x^{(1)})dv + z^{(1)}(u-r) - z^{(1)}(s-r) \right) \\
& \quad + \int_s^u \sigma(\eta(v-2r))(d(y_{\cdot-r} \otimes y)_{\cdot,u}(v) - d(y_{\cdot-r} \otimes y)_{\cdot,t}(v)).
\end{aligned}$$

So

$$\begin{aligned}
& (x_{\cdot-r}^{(1)} \otimes y)_{s,u} + (x_{\cdot-r}^{(1)} \otimes y)_{u,t} \\
& = (x_{\cdot-r}^{(1)} \otimes y)_{s,t} + (y(u) - y(t)) \left(\int_s^u b(v-r, x^{(1)})dv + z^{(1)}(u-r) - z^{(1)}(s-r) \right) \\
& \quad + \int_s^u \sigma(\eta(v-2r))(d(y_{\cdot-r} \otimes y)_{\cdot,u}(v) - d(y_{\cdot-r} \otimes y)_{\cdot,t}(v))
\end{aligned} \tag{4.12}$$

On the other hand, from Definition 2.1 we obtain that

$$\int_s^u \sigma(\eta(v-2r))(d(y_{\cdot-r} \otimes y)_{\cdot,u}(v) - d(y_{\cdot-r} \otimes y)_{\cdot,t}(v)) = (y(u) - y(t)) \int_s^u \sigma(\eta(v-2r))dy_{v-r}. \tag{4.13}$$

Finally, using that

$$\begin{aligned}
\int_s^u b(v-r, x^{(1)})dv &= \int_{s-r}^{u-r} b(v, x^{(1)})dv, \\
\int_s^u \sigma(\eta(v-2r))dy_{v-r} &= \int_{s-r}^{u-r} \sigma(\eta(v-r))dy_v,
\end{aligned}$$

and putting together (4.12) and (4.13) we get the multiplicative property (4.11).

c) Case $0 \leq s \leq r$ and $r \leq u \leq t \leq 2r$.

Notice first that from the definition of $(x_{\cdot-r}^{(1)} \otimes y)$ it follows that

$$(x_{\cdot-r}^{(1)} \otimes y)_{s,u} = (\eta_{\cdot-r} \otimes y)_{s,r} + (x_{\cdot-r}^{(1)} \otimes y)_{r,u} + (\eta(0) - \eta(s-r)) \otimes (y(u) - y(r)). \tag{4.14}$$

On the other hand, we have seen in the case b) (choosing $s = r$) that

$$(x_{\cdot-r}^{(1)} \otimes y)_{r,t} = (x_{\cdot-r}^{(1)} \otimes y)_{r,u} + (x_{\cdot-r}^{(1)} \otimes y)_{u,t} + (x^{(1)}(u-r) - \eta(0)) \otimes (y(t) - y(u)). \tag{4.15}$$

So, putting together (4.14) and (4.15) we can write

$$\begin{aligned}
& (x_{\cdot, -r}^{(1)} \otimes y)_{s,u} + (x_{\cdot, -r}^{(1)} \otimes y)_{u,t} + (x^{(1)}(u-r) - \eta(s-r)) \otimes (y(t) - y(u)) \\
&= (\eta_{\cdot, -r} \otimes y)_{s,r} + (x_{\cdot, -r}^{(1)} \otimes y)_{r,t} + (\eta_0 - \eta_{s-r}) \otimes (y_t - y_r) \\
&= (x_{\cdot, -r}^{(1)} \otimes y)_{s,t},
\end{aligned}$$

where the last equality follows from the definition of $(x_{\cdot, -r}^{(1)} \otimes y)$. The proof of this case is now finished.

d) Case $0 \leq s \leq u \leq r$ and $r \leq t \leq 2r$.

This case can be done following the same ideas that the case c).

4. Now only remains to prove that $|(x_{\cdot, -r}^{(1)} \otimes y)_{s,t}| \leq C|t-s|^{2\beta}$. We will distinguish again three cases:

(a) Assume that $s, t \in [r, 2r]$. Then

$$\begin{aligned}
|(x_{\cdot, -r}^{(1)} \otimes y)_{s,t}| &\leq \left| \int_s^t (y(t) - y(v))b(v-r, x^{(1)})dv \right| + \left| \int_s^t \sigma(\eta(v-2r))d(y_{\cdot, -r} \otimes y)_{\cdot, t}(v) \right| \\
&\quad + \left| \int_s^t (y(t) - y(v))dz_{v-r}^{(1)} \right|.
\end{aligned}$$

Since y is β -Hölder continuous function, we have that

$$\left| \int_s^t (y(t) - y(v))dz_{v-r}^{(1)} \right| \leq K|t-s|^\beta |z^{(1)}(t-r) - z^{(1)}(s-r)|,$$

for a constant K . Then, using Lemma 7.2 we get

$$\left| \int_s^t (y(t) - y(v))dz_{v-r}^{(1)} \right| \leq K|t-s|^{2\beta}. \quad (4.16)$$

On the other hand, using the hypothesis on b we have

$$\begin{aligned}
\left| \int_s^t (y(t) - y(v))b(v-r, x^{(1)})dv \right| &\leq K|t-s|^\beta \left| \int_s^t (L_0 \sup_{-r \leq u \leq v-r} |x^{(1)}(u)| + b_0(v))dv \right| \\
&\leq K|t-s|^{\beta+1} \|x^{(1)}\|_{\infty(-r, r)} + |t-s|^{\beta+1-\frac{1}{\rho}} \|b_0\|_{L^\rho}. \quad (4.17)
\end{aligned}$$

Finally using Proposition 3.4 we get

$$\begin{aligned}
\left| \int_s^t \sigma(\eta(v-2r))d(y_{\cdot, -r} \otimes y)_{\cdot, t}(v) \right| &\leq k|\sigma(\eta(s-2r))| \Phi_{s,t,\beta}(y_{\cdot, -r}, y)(t-s)^{2\beta} \\
&\quad + k \left(\|\sigma'\|_\infty + \|\sigma'\|_\gamma \|\eta_{\cdot, -2r}\|_{\beta(s,t)}^\gamma (t-s)^{\gamma\beta} \right) \Phi_{s,t,\beta}(\eta_{\cdot, -2r}, y_{\cdot, -r}, y)(t-s)^{3\beta}, \quad (4.18)
\end{aligned}$$

where

$$\begin{aligned}
\Phi_{a,b,\beta}(x, y, z) &= \|y\|_{\beta(a,b)} \|z\|_{\beta(a,b)} \|x\|_{\beta(a,b)} \\
&\quad + \|z\|_{\beta(a,b)} \|x \otimes y\|_{2\beta(a,b)} + \|x\|_{\beta(a,b)} \|y \otimes z\|_{2\beta(a,b)}.
\end{aligned}$$

Now putting together (4.16), (4.17) and (4.18) we finish the proof.

(b) If $s \in [0, r]$ and $t \in [r, 2r]$,

$$\begin{aligned} |(x_{\cdot-r}^{(1)} \otimes y)_{s,t}| &\leq |(\eta_{\cdot-r} \otimes y)_{s,r}| + |(x_{\cdot-r}^{(1)} \otimes y)_{r,t}| + |(\eta(0) - \eta(s-r)) \otimes (y(t) - y(r))| \\ &\leq K|r-s|^{2\beta} + K|t-r|^{2\beta} + K|s-r|^\beta |t-r|^\beta \leq K|t-s|^{2\beta}. \end{aligned}$$

(c) If $s, t \in [0, r]$ then $x_{\cdot-r}^{(1)} = \eta_{\cdot-r}$ and the result is already true.

□

5 Boundedness for deterministic integral equations

The aim of this section is the proof of Theorem 2.3. For simplicity let us assume $T = Mr$.

Proof of Theorem 2.3: The proof will be done in several steps.

Step 1: Assuming that $(x, y, x \otimes y) \in M_{d,m}^\beta(0, T)$, let us define $(x_{\cdot-r} \otimes y_{\cdot-r})_{s,t}$. Set

$$(x_{\cdot-r} \otimes y_{\cdot-r})_{s,t} := (x \otimes y)_{s-r, t-r}. \quad (5.1)$$

It clearly belongs to $M_{d,m}^\beta(r, T)$. Notice that the functions $x_{\cdot-r}$ and $y_{\cdot-r}$ are β -Hölder continuous and $x_{\cdot-r} \otimes y_{\cdot-r}$ is a continuous functions satisfying the multiplicative property. Indeed, we have, for $s \leq u \leq t$,

$$\begin{aligned} (x_{\cdot-r} \otimes y_{\cdot-r})_{s,u} + (x_{\cdot-r} \otimes y_{\cdot-r})_{u,t} + (x(u-r) - x(s-r)) \otimes (y(t-r) - y(u-r)) \\ = (x \otimes y)_{s-r, u-r} + (x \otimes y)_{u-r, t-r} + (x(u-r) - x(s-r)) \otimes (y(t-r) - y(u-r)) \\ = (x \otimes y)_{s-r, t-r} = (x_{\cdot-r} \otimes y_{\cdot-r})_{s,t}. \end{aligned}$$

Finally, we also have that, for all $(s, t) \in \{(s, t) : r \leq s < t \leq T\}$,

$$|(x_{\cdot-r} \otimes y_{\cdot-r})_{s,t}| = |(x \otimes y)_{s-r, t-r}| \leq c|t-s|^{2\beta}.$$

Step 2: Set, for any $s, t \in [nr, (n+1)r]$, $n = 0, \dots, (M-1)$,

$$\begin{aligned} J_1^n(x, y, x \otimes y)(t) &= \eta(0) + \int_0^t b(s, x) ds + \int_0^t \sigma(x(s-r)) dy_s + z(t), \\ J_2^n(x, y, x \otimes y)(s, t) &= \begin{cases} (\eta_{\cdot-r} \otimes y)_{s,t}, & \text{if } s, t \in [0, r] \text{ and } n = 1, \\ (x_{\cdot-r} \otimes y)_{s,t}, & \text{otherwise,} \end{cases} \end{aligned}$$

where

$$\begin{aligned} (x_{\cdot-r} \otimes y)_{s,t} &= \int_s^t (y(t) - y(u)) b(u-r, x) du + \int_s^t \sigma(x(u-2r)) d(y_{\cdot-r} \otimes y)_{\cdot,t}(u) \\ &\quad + \int_s^t (y(t) - y(u)) dz_{u-r}, \end{aligned}$$

with z given in (2.1).

Set

$$\Delta_y = \left(\|\eta\|_\beta + \|\eta_{\cdot-r} \otimes y\|_{2\beta} + (d^{\frac{1}{2}} + 1)(1 + 3k)\mu \left[\|y\|_\beta + \|y\|_\beta^2 + \|y_{\cdot-r} \otimes y\|_{2\beta} \right] \right)^{-\frac{1}{\beta}},$$

with k depending only on β and γ . If we consider s, t such that

$$0 < t-s \leq \Delta_y \wedge 1,$$

then we will have that

$$(t-s)^\beta \leq \Delta_y^\beta \leq \frac{1}{\|\eta\|_\beta}, \quad (5.2)$$

$$(t-s)^\beta \leq \Delta_y^\beta \leq \frac{1}{\|\eta_{\cdot-r} \otimes y\|_{2\beta}}, \quad (5.3)$$

$$(t-s)^\beta \leq \Delta_y^\beta \leq \frac{1}{(d^{\frac{1}{2}}+1)(1+3k) \mu \left(1 + \|y\|_\beta + \|y\|_\beta^2 + \|\eta_{\cdot-r} \otimes y\|_{2\beta}\right)}. \quad (5.4)$$

We will use an induction argument to prove that for any n and for all s, t such that $nr \leq s < t \leq (n+1)r$ with $0 < t-s \leq \Delta_y \wedge 1$, it holds that

$$\|J_1^n\|_{\beta(s,t)} = \|x\|_{\beta(s,t)} \leq \mu(d^{\frac{1}{2}}+1) \left[1 + 2k + 3k \|y\|_\beta\right].$$

Assuming this last inequality, the proof of (2.2) is standard. Indeed, notice that it follows easily, that for any $0 \leq s < t \leq T$ with $t-s \leq \Delta_y$,

$$\|x\|_{\beta(s,t)} \leq 2\mu(d^{\frac{1}{2}}+1) \left[1 + 2k + 3k \|y\|_\beta\right].$$

Then, if $t-s \leq \Delta_y$, $0 \leq s < t$, we have

$$\sup_{r \in [s,t]} |x(r)| \leq |x(s)| + (t-s)^\beta \|x\|_{\beta(s,t)} \leq |x(s)| + 2, \quad (5.5)$$

and, particularly,

$$\sup_{r \in [0, \Delta_y]} |x(r)| \leq \eta(0) + 2.$$

Now we divide the interval $[0, T]$ into $n = \lceil T\Delta_y^{-1} \rceil + 1$ intervals of length Δ_y , where $[a]$ denotes the largest integer bounded by a . Then, applying (5.5) on the intervals $[\Delta_y, 2\Delta_y], \dots, [(n-1)\Delta_y, n\Delta_y]$ and the previous inequality, we obtain

$$\sup_{r \in [0, T]} |x(r)| \leq \eta(0) + 2n \leq 2 + \eta(0) + 2 \lceil T\Delta_y^{-1} \rceil,$$

and we can conclude that the estimate (2.2) is true.

Let us come back to check our induction argument to finish the proof.

Step 2.1: Assume $s, t \in [0, r]$. On the one hand,

$$\|J_2^0\|_{2\beta(s,t)} = \|\eta_{\cdot-r} \otimes y\|_{2\beta(s,t)}.$$

On the other hand, using Lemma 7.2 and Proposition 3.3, we obtain that

$$\begin{aligned} \|J_1^0\|_{\beta(s,t)} &\leq (d^{\frac{1}{2}}+1) \left[\|b\|_\infty + k \left(\|\sigma\|_\infty \|y\|_\beta + \left[\|\eta_{\cdot-r} \otimes y\|_{2\beta(s,t)} + \|\eta_{\cdot-r}\|_{\beta(s,t)} \|y\|_\beta \right] \right. \right. \\ &\quad \left. \left. \times \left[\|\sigma'\|_\infty + \|\sigma'\|_\gamma \|\eta_{\cdot-r}\|_{\beta(s,t)}^\gamma (t-s)^{\gamma\beta} \right] (t-s)^\beta \right) \right]. \end{aligned}$$

By means of (5.2) and (5.3), we have

$$\begin{aligned} \|\eta_{\cdot-r}\|_{\beta(s,t)} (t-s)^\beta &\leq 1, \\ \|\eta_{\cdot-r} \otimes y\|_{2\beta(s,t)} (t-s)^\beta &\leq 1. \end{aligned}$$

Then we conclude that

$$\|J_1^0\|_{\beta(s,t)} \leq \mu(d^{\frac{1}{2}} + 1) \left[1 + 2k + 3k \|y\|_{\beta} \right]. \quad (5.6)$$

Step 2.2: Since the first step does not follow the general case, in order to prove our induction we have also to consider the next interval.

So, assume now $s, t \in [r, 2r]$. We first need to deal with $\|J_2^1\|_{2\beta(s,t)} = \|x_{\cdot-r} \otimes y\|_{2\beta(s,t)}$. Applying Proposition 3.4 and dealing with the integrals with b and z , we get

$$\begin{aligned} \|J_2^1\|_{2\beta(s,t)} &\leq \|y\|_{\beta} \|z\|_{\beta(s-r,t-r)} + \mu \left[\|y\|_{\beta} + k \left(\|y_{\cdot-r} \otimes y\|_{2\beta} + \|y\|_{\beta}^2 \right. \right. \\ &\quad \left. \left. + \left[1 + \|x_{\cdot-2r}\|_{\beta(s,t)}^{\gamma} (t-s)^{\gamma\beta} \right] \left[\|x_{\cdot-2r}\|_{\beta(s,t)} \|y\|_{\beta}^2 \right. \right. \right. \\ &\quad \left. \left. \left. + \|x_{\cdot-2r} \otimes y_{\cdot-r}\|_{2\beta(s,t)} \|y\|_{\beta} + \|x_{\cdot-2r}\|_{\beta(s,t)} \|y_{\cdot-r} \otimes y\|_{2\beta} \right] (t-s)^{\beta} \right) \right]. \end{aligned}$$

The same computations given to bound J_1^0 in (5.6) and Lemma 7.2 yields that

$$\|z\|_{\beta(s-r,t-r)} \leq d^{\frac{1}{2}} \mu \left[1 + 2k + 3k \|y\|_{\beta} \right].$$

As before, the estimate (5.2) gives

$$\|x_{\cdot-2r}\|_{\beta(s,t)} (t-s)^{\beta} = \|\eta_{\cdot-2r}\|_{\beta(s,t)} (t-s)^{\beta} \leq 1,$$

and (5.1) and (5.3) imply

$$\|x_{\cdot-2r} \otimes y_{\cdot-r}\|_{2\beta(s,t)} (t-s)^{\beta} = \|\eta_{\cdot-r} \otimes y\|_{2\beta(s-r,t-r)} (t-s)^{\beta} \leq 1.$$

So, using all these inequalities, we can bound J_2^1 as follows:

$$\begin{aligned} \|J_2^1\|_{2\beta(s,t)} &\leq (d^{\frac{1}{2}} + 1)(1 + 2k)\mu \|y\|_{\beta} + (d^{\frac{1}{2}} + 1)3k\mu \|y\|_{\beta}^2 + 3k\mu \|y_{\cdot-r} \otimes y\|_{2\beta} \\ &\leq (d^{\frac{1}{2}} + 1)(1 + 3k)\mu \left(\|y\|_{\beta} + \|y\|_{\beta}^2 + \|y_{\cdot-r} \otimes y\|_{2\beta} \right). \end{aligned} \quad (5.7)$$

Proposition 3.3 again yields

$$\begin{aligned} \|J_1^1\|_{\beta(s,t)} &\leq (d^{\frac{1}{2}} + 1) \left[\|b\|_{\infty} + k \left(\|\sigma\|_{\infty} \|y\|_{\beta} + \left[\|x_{\cdot-r} \otimes y\|_{2\beta(s,t)} + \|x_{\cdot-r}\|_{\beta(s,t)} \|y\|_{\beta} \right] \right. \right. \\ &\quad \left. \left. \times \left[\|\sigma'\|_{\infty} + \|\sigma'\|_{\gamma} \|x_{\cdot-r}\|_{\beta(s,t)}^{\gamma} (t-s)^{\gamma\beta} \right] (t-s)^{\beta} \right) \right]. \end{aligned}$$

Now, using the computations that we have done in the first step

$$\|x_{\cdot-r}\|_{\beta(s,t)} = \|J_1^0\|_{\beta(s-r,t-r)} \leq \mu(d^{\frac{1}{2}} + 1) \left[1 + 2k + 3k \|y\|_{\beta} \right],$$

and (5.4) yields that

$$\|x_{\cdot-r}\|_{\beta(s,t)} (t-s)^{\beta} \leq 1.$$

Moreover, also thanks to (5.7) and (5.4)

$$\|x_{\cdot-r} \otimes y\|_{2\beta(s,t)} (t-s)^{\beta} = \|J_2^1\|_{2\beta(s,t)} (t-s)^{\beta} \leq 1.$$

Combining these two last bounds allow us to show that

$$\|J_1^1\|_{\beta(s,t)} \leq \mu(d^{\frac{1}{2}} + 1) \left[1 + 2k + 3k \|y\|_{\beta} \right]. \quad (5.8)$$

Step 2.3: We can write now our hypothesis of induction. Set (H_l) : for any $s, t \in [lr, (l+1)r]$, the following hypothesis are satisfied

$$(H_l) \quad \begin{cases} \|J_1^l\|_{\beta(s,t)} \leq \mu(d^{\frac{1}{2}} + 1) \left[1 + 2k + 3k \|y\|_{\beta} \right], \\ \|J_2^l\|_{2\beta(s,t)} \leq \mu(d^{\frac{1}{2}} + 1)(1 + 3k) \left[\|y\|_{\beta} + \|y\|_{\beta}^2 + \|y_{\cdot-r} \otimes y\|_{2\beta} \right]. \end{cases}$$

We have checked (H_1) . Let us suppose that (H_l) is satisfied for any $l = 1, \dots, n$. Our goal is to prove that (H_{n+1}) is also satisfied. For $s, t \in [(n+1)r, (n+2)r]$, the first two inequalities can be proved as in the previous case:

$$\begin{aligned} \|J_1^{n+1}\|_{\beta(s,t)} &\leq (d^{\frac{1}{2}} + 1)\mu \left[1 + k \left(\|y\|_{\beta} + \|x_{\cdot-r} \otimes y\|_{2\beta(s,t)} + \|x_{\cdot-r}\|_{\beta(s,t)} \|y\|_{\beta} \right) \right. \\ &\quad \left. \times \left[1 + \|x_{\cdot-r}\|_{\beta(s,t)}^{\gamma} (t-s)^{\gamma\beta} \right] (t-s)^{\beta} \right], \\ \|J_2^{n+1}\|_{2\beta(s,t)} &= \|x_{\cdot-r} \otimes y\|_{2\beta(s,t)} \\ &\leq \|y\|_{\beta} \|z\|_{\beta(s-r, t-r)} + \mu \left[\|y\|_{\beta} + k \left(\|y_{\cdot-r} \otimes y\|_{2\beta} + \|y\|_{\beta}^2 \right. \right. \\ &\quad \left. \left. + \left[1 + \|x_{\cdot-2r}\|_{\beta(s,t)}^{\gamma} (t-s)^{\gamma\beta} \right] \right. \right. \\ &\quad \left. \left. \times \left[\|x_{\cdot-2r}\|_{\beta(s,t)} \|y\|_{\beta}^2 + \|x_{\cdot-2r} \otimes y_{\cdot-r}\|_{2\beta(s,t)} \|y\|_{\beta} \right. \right. \right. \\ &\quad \left. \left. \left. + \|x_{\cdot-2r}\|_{\beta(s,t)} \|y_{\cdot-r} \otimes y\|_{2\beta} \right] (t-s)^{\beta} \right] \right]. \end{aligned}$$

In order to study J_1^{n+1} and J_2^{n+1} , we will use that

$$\begin{aligned} \|z\|_{\beta(s-r, t-r)} &\leq d^{\frac{1}{2}} \mu \left[1 + 2k + 3k \|y\|_{\beta} \right], \\ \|x_{\cdot-2r}\|_{\beta(s,t)} (t-s)^{\beta} &= \|J_1^{n-1}\|_{\beta(s-2r, t-2r)} (t-s)^{\beta} \leq 1, \\ \|x_{\cdot-2r} \otimes y_{\cdot-r}\|_{2\beta(s,t)} (t-s)^{\beta} &= \|x_{\cdot-r} \otimes y\|_{2\beta(s-r, t-r)} (t-s)^{\beta} \\ &= \|J_2^n\|_{2\beta(s-r, t-r)} (t-s)^{\beta} \leq 1, \\ \|x_{\cdot-r}\|_{\beta(s,t)} (t-s)^{\beta} &= \|J_1^n\|_{\beta(s-r, t-r)} (t-s)^{\beta} \leq 1. \end{aligned}$$

Then, it is not difficult to check that

$$\begin{aligned} \|J_2^{n+1}\|_{2\beta(s,t)} &\leq \mu(d^{\frac{1}{2}} + 1)(1 + 3k) \left[\|y\|_{\beta} + \|y\|_{\beta}^2 + \|y \otimes y\|_{2\beta} \right], \\ \|J_1^{n+1}\|_{\beta(s,t)} &\leq \mu(d^{\frac{1}{2}} + 1) \left[1 + 2k + 3k \|y\|_{\beta} \right], \end{aligned}$$

where the bound of J_2^{n+1} is used to prove the last inequality. □

6 Stochastic case

Fix a parameter $H \in (\frac{1}{3}, \frac{1}{2})$. Set $W^H = \{W^H(t), t \in [0, T]\}$ a m -dimensional fractional Brownian motion of Hurst parameter H . The components $W^{H,1}, \dots, W^{H,m}$ are independent centered Gaussian processes with the covariance function

$$R(t, s) = \frac{1}{2} (s^{2H} + t^{2H} - |t-s|^{2H}).$$

Let us consider now a Stratonovich type integral with respect to W^H . Following the approach by Russo and Vallois [20], we have:

Definition 6.1 *Let $u = \{u(t), t \in [0, T]\}$ be a stochastic process with integrable trajectories. The Stratonovich integral of u with respect to $W^{H,i}$ is defined as the limit in probability as ε tends to zero of*

$$(2\varepsilon)^{-1} \int_0^T u(s)(W_{s+\varepsilon}^{H,i} - W_{s-\varepsilon}^{H,i})ds,$$

provided this limit exists. When the limit exists, it is denoted by $\int_0^T u(t) \circ dW_t^{H,i}$.

Then, we can consider the tensor product defined by

$$(W_{\cdot-r}^{H,i} \otimes W^{H,j})_{s,t} = \int_s^t (W_{v-r}^{H,i} - W_{s-r}^{H,i}) \circ dW_v^{H,j},$$

for $0 \leq s < t \leq T$ and $i, j = 1, \dots, m$. Notice that when $i \neq j$, the Stratonovich integral coincides with the Skorohod integral.

It is proved in Proposition 5.2 in [18] that

$$E(|(W_{\cdot-r}^{H,i} \otimes W^{H,j})_{s,t}|^p) \leq C_p |t-s|^{2pH}, \quad (5.1)$$

for all $i, j \in \{1, \dots, m\}$ and $p \geq 1$. Furthermore, they also have checked that there is a version of $(W_{\cdot-r}^H \otimes W^H)$ such that, for almost all sample paths of W^H , satisfy that $(W_{\cdot-r}^{H,i} \otimes W^{H,j})_{s,t} \in \mathcal{C}_2^{2\beta}(\mathbb{R}^{m \times m})$ for any $\beta \in (\frac{1}{3}, H)$ and $i, j \in \{1, \dots, m\}$, where $\mathcal{C}_2^{2\beta}$ denotes a space of 2β -Hölder continuous functions of two variables.

Since the multiplicative property can be checked easily using the definition and the properties of the Stratonovich integral, it follows that $(W_{\cdot-r}^H, W^H, W_{\cdot-r}^H \otimes W^H)$ is a β -Hölder continuous multiplicative functional for a fixed $\beta \in (\frac{1}{3}, H)$.

Notice also that (5.1) implies that

$$E(\|W_{\cdot-r}^{H,i} \otimes W^{H,j}\|_{2\beta(r,T)}^p) \leq C_p |t-s|^{2p(H-\beta)},$$

for any $\beta \in (\frac{1}{3}, H)$ and for all $p \geq 1$.

Then Theorem 2.4 follows easily from Theorems 2.2 and Theorem 2.3, when we apply its results pathwise.

7 Appendix

Let us recall a fixed point Theorem from [2].

Lemma 7.1 *Let (X, ρ) be a complete metric space, and ρ_0 and ρ_1 two metrics on X equivalent to ρ . If $\mathcal{L} : X \rightarrow X$ satisfies:*

1. *There exists $r_0 > 0$, $x_0 \in X$ such that if $B_0 = \{x \in X; \rho_0(x_0, x) \leq r_0\}$ then $\mathcal{L}(B_0) \subseteq B_0$,*
2. *There exists $a \in (0, 1)$ such that $\rho_1(\mathcal{L}(x), \mathcal{L}(y)) \leq a\rho_1(x, y)$ for all $x, y \in B_0$.*

Then there exists $x^ \in \mathcal{L}(B_0) \subseteq X$ such that $x^* = \mathcal{L}(x^*)$.*

We also need a result with some properties of the solution of Skorohod's problem.

Lemma 7.2 For each path $\xi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$, there exists a unique solution (x, z) to the Skorokhod problem for ξ . Thus there exists a pair of functions $(\phi, \varphi) : \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d) \rightarrow \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^{2d})$ defined by $(\phi(\xi), \varphi(\xi)) = (x, z)$. The pair (ϕ, φ) satisfies the following:

There exists a constant $K_l > 0$ such that for any $\xi_1, \xi_2 \in \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d)$ we have for each $t \geq 0$,

$$\begin{aligned} \|\phi(\xi_1) - \phi(\xi_2)\|_{\infty(0,t)} &\leq K_l \|\xi_1 - \xi_2\|_{\infty(0,t)}, \\ \|\varphi(\xi_1) - \varphi(\xi_2)\|_{\infty(0,t)} &\leq K_l \|\xi_1 - \xi_2\|_{\infty(0,t)}. \end{aligned}$$

Moreover for each $0 \leq s < t \leq T$

$$\|\varphi(\xi)\|_{\beta(s,t)} \leq d^{\frac{1}{2}} \|\xi\|_{\beta(s,t)},$$

Proof: We refer the reader to Proposition A.0.1 in [11] for the proof of the first part of the Lemma.

Set $z = \varphi(\xi)$. Take u, v such that $s \leq u < v \leq t$. Fixed a component i , we wish to study

$$\frac{|z^i(v) - z^i(u)|}{(v - u)^\beta}.$$

When $z^i(v) = z^i(u)$, this is clearly zero. On the other hand, when $z^i(v) > z^i(u)$, let us define

$$\begin{aligned} u^* &:= \sup\{u' \geq u; z^i(u) = z^i(u')\}, \\ v^* &:= \inf\{v' \leq v; z^i(v) = z^i(v')\}. \end{aligned}$$

Then, $u \leq u^* < v^* \leq v$ and $z^i(u) = z^i(u^*)$, $z^i(v) = z^i(v^*)$. So

$$\frac{|z^i(v) - z^i(u)|}{(v - u)^\beta} \leq \frac{|z^i(v^*) - z^i(u^*)|}{(v^* - u^*)^\beta} = \frac{|\xi^i(v^*) - \xi^i(u^*)|}{(v^* - u^*)^\beta}$$

where the last equality follows from the fact that ξ^i and z^i coincides whenever z^i is not constant.

Then, note that

$$\sup_{s < u < v < t} \frac{|z^i(v) - z^i(u)|}{(v - u)^\beta} \leq \sup_{s < u^* < v^* < t} \frac{|\xi^i(v^*) - \xi^i(u^*)|}{(v^* - u^*)^\beta} \leq \|\xi\|_{\beta(s,t)}.$$

Finally, we get that

$$\|\xi\|_{\beta(s,t)} \leq \left(\sum_{i=1}^d \left(\sup_{s < u < v < t} \frac{|z^i(v) - z^i(u)|}{(v - u)^\beta} \right)^2 \right)^{\frac{1}{2}} \leq d^{\frac{1}{2}} \|\xi\|_{\beta(s,t)}.$$

□

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